2022 Online Physics Olympiad: Invitational Contest

Theoretical Examination

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Instructions for Theoretical Exam

The theoretical examination consists of 5 long answer questions and 110 points over 2 full days from July 30, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

- **T1: Maxwell's Demon** by Zhening Li
- **T2: Euler's Disk** by Daniel Seungmin Lee
- **T3: Rocket** by Adithya Balachandran
- **T4: Magical Box** by Daniel Seungmin Lee
- **T5: Quantum Computing** by QiLin Xue

List of Constants

Proton mass	$m_p = 1.67 \cdot 10^{-27}$ kg
Neutron mass	$m_n = 1.67 \cdot 10^{-27}$ kg
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg
Avogadro's constant	$N_0 = 6.02 \cdot 10^{23}\ \text{mol}^{-1}$
Universal gas constant	$R = 8.31\ \text{J}/(\text{mol} \cdot \text{K})$
Boltzmann's constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Electron charge magnitude	$e = 1.60 \cdot 10^{-19}$
1 electron volt	$1\ \text{eV} = 1.60 \cdot 10^{-19}$ J
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Universal Gravitational constant	$G = 6.67 \cdot 10^{-11}\ (\text{N} \cdot \text{m}^2)/\text{kg}^2$
Acceleration due to gravity	$g = 9.81\ \text{m/s}^2$
1 unified atomic mass unit	$1\ \text{u} = 1.66 \cdot 10^{-27}\ \text{kg} = 931\ \text{MeV/c}^2$
Planck's constant	$h = 6.63 \cdot 10^{-34}\ \text{J} \cdot \text{s} = 4.41 \cdot 10^{-15}\ \text{eV} \cdot \text{s}$
Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}\ \text{C}^2/(\text{N} \cdot \text{m}^2)$
Coulomb's law constant	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\text{N} \cdot \text{m}^2)/\text{C}^2$
Permeability of free space	$\mu_0 = 4\pi \cdot 10^{-7}\ \text{T} \cdot \text{m/A}$
Stefan-Boltzmann constant	$\sigma = 5.67 \cdot 10^{-8}\ \text{W/m}^2/\text{K}^4$

T1: Maxwell's Demon

Zed has a container divided by a wall into two chambers of equal volume V. The left chamber has N_1 molecules and the right chamber has N_2 molecules of some monatomic ideal gas ($N_1 < N_2$). Each gas molecule has mass m and can be treated as a point particle. The entire system is isolated and is at temperature T.

(a) **(5 pts.)** Let's say that he makes a hole in the wall. Then there will be a net flow of molecules from the right chamber to the left chamber. At equilibrium, let's say each chamber has $N = (N_1 + N_2)/2$ molecules. By how much has the entropy increased?

Zed now wants to revert the container back to its original state with N_1 and N_2 molecules in each chamber. He plans to achieve this by covering the hole with a door with area A that only opens towards the second chamber.

(b) **(5 pts.)** He thinks that any molecule in the left chamber incident on the door will enter the right chamber, and no molecules in the right chamber will enter the left one. Under such a model, what is the initial rate of change in entropy of the system?

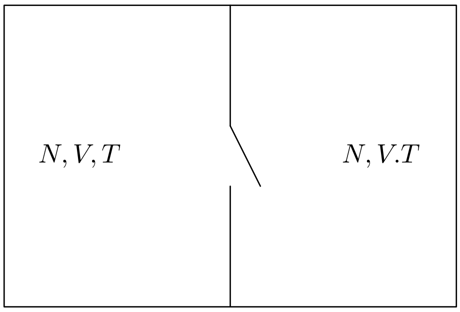


Figure 1: Parts (c) and (d)

Under the assumptions made by part (b), Zed's device violates the second law of thermodynamics. We'll now investigate why this actually does not happen for a particular kind of door. This door, of mass M, has a hinge that exerts a restoring torque $\tau = K\theta$ when the door is open at an angle θ, where θ is not necessarily small (Figure 1).

(c) **(5 pts.)** Explain in one or two sentences why this door behaves effectively like a hole in the wall with area A', and hence the second law of thermodynamics is not violated.

(d) **(10 pts.)** Estimate A' in terms of the variables given and fundamental constants. You may make appropriate simplifying assumptions.

Solution

(a) The internal energy of the system must remain the same as the system is isolated. This implies the temperature of each of the molecules remains as T even when the hole is created. As entropy is a state function and the temperature remains the same, the change in entropy is caused by isothermal expansion/compression. There are two contributions to entropy change: the one experienced by the particles in the first chamber and those in the second chamber.

Since each container as a volume of V, the total volume that each particle can occupy is now $2V$. The number density is thus $\nu = \frac{N_1+N_2}{2V} = \frac{N}{V}$. The particles in the first chamber increase in volume from V to $\frac{V}{\nu}$. This means that

$$\Delta S_1 = N_1 k_B \ln \frac{N_1/\nu}{V} = N_1 k_B \ln \frac{N_1}{N}.$$

Similarly, for particles in the second chamber, we have

$$\Delta S_2 = N_2 k_B \ln \frac{N_2}{N}.$$

The total entropy change is thus

$$\Delta S = \Delta S_1 + \Delta S_2 = (N_1 \ln N_1 + N_2 \ln N_2 - 2 \ln N) k_B$$

Note: Max 2 points for the approach that assumes particles in N_1 are distinguishable from particles in N_2. In this approach, the volume of each compartment doubles so the answer is $(N_1 + N_2) k \ln 2$.

(b) S is a function of N_1. Therefore, we can express $\dot{S}$ as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}N_1} \frac{\mathrm{d}N_1}{\mathrm{d}t}.$$

Right when the door is opened, the system is essentially in equilibrium, so $\frac{\mathrm{d}S}{\mathrm{d}N_1} = 0$. Hence, $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$ as well.

Note: Even though initially $\dot{S} = 0$, it is negative for $t > 0$, and the entropy of the system decreases.

(c) The door is almost always open due to thermal contact with the wall at temperature T. How open the door is doesn't significantly increase as a particle hits it from the left, since its energy is on the same order as the energy of the door $\sim kT$. The effective size of the opening is thus the same for particles traveling in both directions, so particles pass through the opening in both directions at the same rate, hence keeping the entropy constant.

Note: This is just an intuitive explanation as to why we shouldn't find it surprising that there's an equal particle flow in the opposite direction. A rigorous argument (that is not cyclic) would be much more involved.

Marking scheme:

Door is generally open due to thermal motion of molecules in wall	3 pts
Door doesn't get much more open when a particle hits it from the left	2 pts

Explanations that get partial credit:

- When a particle hits the door from the left, the door opens and a particle from the right passes through: max. 3pts (only explains the existence of particle flow in the opposite direction)
- Due to the equipartition theorem, $\frac{1}{2} K \langle \theta^2 \rangle = \frac{1}{2} kT > 0$, so the door is effectively open: max. 3pts (same reason as above)

(d) The equipartition theorem gives $\frac{1}{2} K \langle \theta^2 \rangle = \frac{1}{2} kT$, so the typical $\theta \approx \sqrt{kT/K}$. When $\sqrt{kT/K} 1$, the size of the opening is about $2A \sin\left(\frac{1}{2}\sqrt{\frac{kT}{K}}\right)$; when $\sqrt{kT/K} \gtrsim 1$, the size of the opening is better approximated by the size of the hole, A.

Note: In reality, there's a smooth transition from the $\sqrt{kT/K} 1$ regime to the $\sqrt{kT/K} \gtrsim 1$ regime. A precise model would likely involve modeling the joint distribution $\rho(n, \mathbf{v}, \theta, \dot{\theta} | \mathbf{r})$ of the door's angle and its rate of change, and the particle density and velocity distribution of gas particles at each point in space.

T2: Euler's Disk

A thin, uniform disk of mass m and radius a is initially set at an angle α_0 to the horizontal, on a frictionless surface. It is given an initial angular velocity Ω_0 with respect to a vertical axis passing through its center.

(a) **(4 pts.)** Determine Ω_0 for the steady state case, where $\dot{\alpha} = \ddot{\alpha} = \dot{\Omega} = 0$.

(b) **(2 pts)** Write an expression for the total energy of the disk.

The disk is then moved onto a special surface with small bumps of height h spread over it − each bump is separated by δ. As the disk climbs over a bump and falls back down, its impact is absorbed by the surface, causing a net energy loss in the system. The disk is set in motion with the same initial conditions as before but with $\alpha_0 << 1$

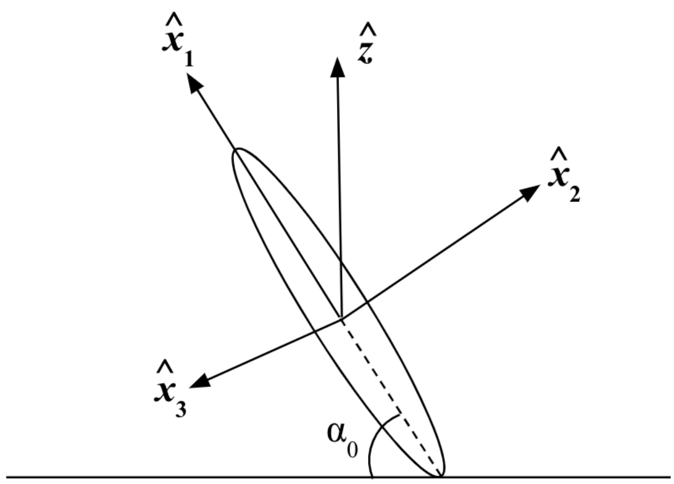
(c) **(6 pts.)** Assuming that this is the only source of energy loss, write a differential equation for $\dot{\alpha}$ in first order to α.

(d) **(4 pts.)** Hence, write an approximate expression for Ω as a function of time.

(e) **(2 pts.)** Using this model, determine the time it takes for the frequency of the sound the disk makes against the surface to reach the maximum audible frequency f_0.

Solution

(a) Let us use the following three axes:



Let's say the disk moves in a counter-clockwise direction when viewed from above. Since the disk is given an angular velocity $\Omega_0\hat{\mathbf{z}}$, it must roll without slipping with angular velocity $-\omega'\,\hat{\mathbf{x}}_2$. Since it rolls without slipping:

$$\frac{2\pi a\cos(\alpha_0)}{\omega' a} = \frac{2\pi}{\Omega_0}$$

Thus,

$$\omega' = \Omega_0\cos(\alpha_0)$$

Therefore, the net angular velocity vector is:

$$\boldsymbol{\omega} = \Omega_0\hat{\mathbf{z}} - \omega'\,\hat{\mathbf{x}}_2 = \Omega_0\hat{\mathbf{z}} - \Omega_0\cos(\alpha_0)\hat{\mathbf{x}}_2 = \Omega_0\sin(\alpha_0)\hat{\mathbf{x}}_1$$

Since the disk's point of contact with the ground and the COM is at rest, we can let the axis passing through both points – $\hat{\mathbf{x}}_\mathbf{1}$ – be the instantaneous axis of rotation. Note that this is validated by the net angular velocity vector lying along $\hat{\mathbf{x}}_\mathbf{1}$. By the perpendicular axes theorem, the moment of inertia around $\hat{\mathbf{x}}_\mathbf{1}$ is $I = \frac{1}{4}ma^2$. Thus, the angular momentum vector is:

$$\mathbf{L} = I\boldsymbol{\omega} = \frac{\Omega_0}{4}ma^2\sin(\alpha_0)\hat{\mathbf{x}}_\mathbf{1}$$

If we calculate torque $\boldsymbol{\tau}$ from the point of contact with the ground, the only force that contributes is weight. Thus,

$$\boldsymbol{\tau} = mga\cos(\alpha_0)\hat{\mathbf{x}}_\mathbf{3}$$

This should equal:

$$\frac{d\mathbf{L}}{dt} = \frac{\Omega_0}{4}ma^2\sin(\alpha_0)\frac{d\hat{\mathbf{x}}_\mathbf{1}}{dt}$$

Let's do some quick vector analysis to find $\frac{d\hat{\mathbf{x}}_\mathbf{1}}{dt}$. Note that, since $\hat{\mathbf{x}}_\mathbf{1}$ is the instantaneous axis of rotation, the unit vectors $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{3}$ spin on the plane of the disk at this given instant. Suppose $\hat{\mathbf{x}}_\mathbf{1}$ spins for an infinitesimal time t such that its coordinates on the plane of the disk is:

$$\hat{\mathbf{x}}_\mathbf{1} = \langle\sin(\Omega_0\cos\alpha_0\, t), \cos(\Omega_0\cos\alpha_0\, t)\rangle$$

Taking the time derivative,

$$\frac{d\hat{\mathbf{x}}_\mathbf{1}}{dt} = \Omega_0\cos\alpha_0\,\langle-\cos(\Omega_0\cos\alpha_0\, t), \sin(\Omega_0\cos\alpha_0\, t)\rangle = \Omega_0\cos\alpha_0\hat{\mathbf{x}}_\mathbf{3}$$

Plugging this into the equation above and solving for Ω_0,

$$\Omega_0 = \sqrt{\frac{4g}{a\sin(\alpha_0)}}$$

(b) As derived in the previous question, the total rotational kinetic energy of the disk can be written as:

$$\frac{1}{2}\left(\frac{1}{2}ma^2\right)(\Omega_0\sin\alpha_0)^2$$

again, since $\hat{\mathbf{x}}_\mathbf{1}$ is the IAR. Since the COM of the disk remains at height $H = a\sin\alpha_0$, we can write its total energy as:

$$\frac{1}{2}\left(\frac{1}{2}ma^2\right)(\Omega_0\sin\alpha_0)^2 + mga\sin\alpha_0 = \frac{3}{2}mga\sin\alpha_0$$

(c) We simply have to add a $\frac{1}{2}m\dot{H}^2$ term to the total energy above to take into account the disk's falling COM. Approximating the total energy E at the $\alpha \to 0$ limit,

$$E = \frac{1}{2}ma^2(\dot{\alpha}\cos\alpha)^2 + \frac{3}{2}mga\sin\alpha \approx \frac{1}{2}ma^2\dot{\alpha}^2 + \frac{3}{2}mga\alpha$$

Now, we'll find the power P dissipated by the bumps and equate $P = \frac{dE}{dt}$. As the disk climbs over a bump and falls back down, its gravitational potential energy mgh will be absorbed by the surface. In other words, the disk will lose mgh every $\frac{\delta}{\Omega\cos\alpha a} \approx \frac{\delta}{\Omega a}$. Hence, we get:

$$P = -\frac{mgha\Omega}{\delta}$$

Therefore,

$$ma^2\dot{\alpha}\ddot{\alpha} + \frac{3}{2}mga\dot{\alpha} = -\frac{mgha\Omega}{\delta}$$

Assuming the disk spends most of its time falling down from a bump, we can approximate its COM's acceleration to be $a\ddot{\alpha} \approx g$. Fully simplifying the equation above, we get:

$$\dot{\alpha} = -\frac{4h}{5\delta}\sqrt{\frac{g}{a}}\frac{1}{\sqrt{\alpha}}$$

(d) Solving the differential equation from the previous question gives

$$\alpha^{\frac{3}{2}} - \alpha_0^{\frac{3}{2}} = -\frac{6h}{5\delta}\sqrt{\frac{g}{a}}t$$

We can simply plug in α into $\Omega = \sqrt{\frac{4g}{a\alpha}}$ to find the answer.

(e) The disk will produce a "click" every time it falls off a bump, hitting the surface. The frequency f at any given time will therefore be:

$$f = \frac{\delta}{\Omega a}$$

Any attempt with this idea were given full marks.

T3: Rocket

OPhO organizers have a "propulsionless" rocket, which for simplicity can be assumed to be a 2-dimensional rectangular box of mass $2M$ and horizontal length L. Assume that the horizontal sides of the box are massless while the vertical sides of the box each have mass M. The rocket is initially at rest. We will now explore the mechanism for how this rocket move. Suppose we have N particles of mass m/N each on the left and right sides of the box. At time $t = 0$, we launch the N particles on the left side of the box together to the right with velocity $\frac{v}{N}$. In addition, in intervals of time $\frac{L}{v}$, starting at $t = 0$, we launch a particle from the right side of the box to the left side with velocity v. Once a particle reaches the opposite side of the box, it is stopped. The particular mechanism to shoot and catch the particles can be ignored here. Assume that this mechanism can conserve energy. After time $t = \frac{NL}{v}$, there will be N particles on each side of the box, which is identical to the initial state.

Neglect relativistic effects in part (a) only.

(a) **(1 pt.)** According to classical (Newtonian) mechanics, what happens to the rocket? Does it move?

(b) **(5 pts.)** If $v \ll c$, how far does the rocket move? Answer in lowest nonzero order in v/c.

(c) **(10 pts.)** How far does the center of mass of the rocket system move? Once again, answer in lowest nonzero order in v/c. Justify your answer.

(d) **(6 pts.)** Explain why this process cannot continue indefinitely. If it could continue forever, we would able to move the rocket indefinitely with no propulsion.

(e) **(5 pts.)** Give an estimate for how long this process can continue. How far does the rocket move in this time?

Solution

(a) At each moment, the particles from the left side of the rocket have momentum $N\frac{m}{N}\frac{v}{N}$ and the particles from the right side of the box have momentum $\frac{mv}{N}$. As these are equal, by conservation of momentum for the entire system, the rocket must have 0 momentum, so it does not move.

(b) Let $\beta = v/c$. The relativistic momentum of the particles from the left side of the rocket is

$$N\frac{1}{\sqrt{1-\frac{\beta^2}{N^2}}}\frac{m}{N}\frac{\beta c}{N} = \frac{\beta mc}{N\sqrt{1-\beta^2/N^2}}.$$

The momentum of the particle from the right side of the rocket is

$$\frac{1}{\sqrt{1-\beta^2}}\frac{m}{N}(\beta c) = \frac{\beta mc}{N\sqrt{1-\beta^2}}.$$

The difference in these quantities to lowest nonzero order in β is

$$\frac{\beta mc}{N}\left(1+\frac{\beta^2}{2}-1-\frac{\beta^2}{2N^2}\right) = \frac{\beta^3 mc(N^2-1)}{2N^3}.$$

This is the momentum of the rocket to the right. Thus, it is equal to $\gamma(2M + \frac{N-1}{N}m)V$ if V is the speed of the rocket. As the momentum is third order in β, we can assume that $\gamma \approx 1$ to find V to lowest nonzero order. We obtain

$$V = \frac{\beta^3 mc(N^2-1)}{2N^2(2MN+(N-1)m)}.$$

Now, the time is $t = \frac{NL}{v} = \frac{NL}{\beta c}$, so the total distance traveled is

$$Vt = \frac{(N^2-1)m}{2N(2MN+(N-1)m)}\frac{v^2}{c^2}L.$$

Valid solutions that assume $N \gg 1$ or $m \ll M$ are acceptable.

(c) The center of energy in a closed relativistic system with zero total momentum does not move, so the rocket's center of mass at the end is exactly where it was in the beginning.

For a proof of this fact, consider a system of particles. We wish to show the claim that $\sum_i x_i E_i$ is constant. Whenever there is a collision between particles i and j, note that just before and after the collision, both particles are at the same position x, so if E_i' and E_j' are the energies after the collision, then the change in $\sum_i x_i E_i$ during the collision is $xE_i' + xE_j' - xE_i - xE_j = x(E_i' + E_j' - E_i - E_j) = 0$. This is true as energy is conserved during the collision. We have the same conclusion when one particle decays for the same reason. Lastly, it suffices to show that $\sum_i x_i E_i$ is conserved for a set of particles moving freely with zero total momentum. In this case, E_i is constant and equal to $\gamma_i m_i c^2$, so the change in $\sum_i x_i E_i$ is $\sum_i (v_i t)\gamma_i m_i c^2 = c^2 t \sum_i \gamma_i m_i v_i = c^2 t \sum_i p_i = 0$. Here, we have assumed that the particles move in one dimension, but it is simple to extend this to multiple dimensions. Therefore, we can conclude that the center of energy does not move.

Half credit will be given for the answer and a classical explanation. For full credit, a satisfactory relativistic explanation must be provided.

(d) The center of mass does not move although the rocket system moves to the right. This means that through this process, mass is transferred from the right side of the rocket to the left side. Every time this process is repeated, the same amount of mass is transferred from the right to the left side, and since the right side only starts with mass M, the process cannot continue indefinitely. Unfortunately, we cannot construct a propulsionless rocket even with relativity.

(e) If a mass δm is transferred from the left side to the right side in each step, then the center of mass of the rocket moves a distance $\frac{\delta m}{2M}L$ to the left from the geometric center of the rocket. In order for the center of mass of the rocket to not move, we must have

$$\frac{\delta m}{2M}L = \frac{(N^2-1)m}{2N(2M+(N-1)m)}\frac{v^2}{c^2}L,$$

$$\delta m = \frac{(N^2-1)mM}{N(2MN+(N-1)m)}\frac{v^2}{c^2}.$$

The number of times that this can continue is roughly

$$\frac{M}{\delta m} = \frac{N(2MN+(N-1)m)}{(N^2-1)m}\frac{c^2}{v^2}.$$

The total distance that can be traveled is roughly at most $L/2$ (if all of the mass from the right goes to the left).

Here, we implicitly assumed $m \ll M$. Any solution with correct reasoning and a valid estimate is acceptable.

T4: Magical Box

A cubical box of mass M and side length L sits on a horizontal, frictionless plane. The box is filled with an ideal gas of particle mass m, particle volume density n, and initial temperature T_0. One of the vertical walls inside the cube is made of a highly conductive material, kept at a constant temperature $T_b \gg T_0$. The wall is so conductive that the temperature of gas instantaneously changes to T_b after rebounding. All other walls are made of ideal insulators.

(a) **(1 pt.)** State, with a reasoning, the direction in which the box will start moving.

(b) **(7 pts.)** Approximate the initial acceleration a_0 of the box. For this question, make sure your equation is valid for $T_b = T_0$ as well.

(c) **(3 pts.)** The acceleration of the box then decreases from a_0 to a_f for a short time until $t = \tau_0$. Determine a_f.

(d) **(3 pts.)** If τ_1 is the time it takes for acceleration to level off for an identical box with the conductive wall at temperature $\frac{T_b}{3}$, calculate $\frac{\tau_1}{\tau_0}$.

Solution

(a) Let us consider two particles of mass m, both starting at the center of the box. Particle A travels in the x-direction to the conductive wall, while particle B similarly goes to the opposite insulated one, both travelling at a speed v. We neglect the impact of the other walls, as due to symmetry, any effects will cancel out. Furthermore, we can take such an approximation because Maxwell's distribution shows that half the particles would travel to the opposite wall and vice versa.

Due to the given conditions, the momentum of the system must initially be zero. As the velocity of a particle is proportional to the square root of its temperature, $v \propto \sqrt{T}$, then the velocity of A after rebounding will be slightly greater as $v + \Delta v$ in the negative x-direction. The velocity of B after rebounding will be the same in the positive x-direction. As such, the net momentum of both particles is $-m\Delta v$. This means that the box must start moving in the positive x-direction, or in the direction of the conductive wall, to conserve the momentum of the system.

(b) Inside the box, the particles will be moving randomly at different speeds. We want to find the number of particles approaching a wall in the given speeds $[v, v + \mathrm{d}v]$ and angles $[\theta, \theta + \mathrm{d}\theta]$. If all molecules are moving in equal directions, the fraction of particles within a solid angle $\mathrm{d}\Omega$ is $\mathrm{d}\Omega/4\pi$. If we consider the angles between θ and $\theta + \mathrm{d}\theta$, we can relate Ω as

$$\mathrm{d}\Omega = 2\pi \sin\theta \mathrm{d}\theta \implies \frac{\mathrm{d}\Omega}{4\pi} = \frac{1}{2}\sin\theta \mathrm{d}\theta.$$

The number of particles in a unit volume is then

$$\rho = nf(v)\mathrm{d}v\frac{1}{2}\sin\theta \mathrm{d}\theta$$

where $f(v)$ is the Maxwell speed distribution function. For molecules approaching a wall of area A at angle θ, the volume encapsulated within a unit time $\mathrm{d}t$ is

$$\mathrm{d}V = Av\mathrm{d}t\cos\theta$$

where A is the area of the wall, or in this case, L^2. Therefore, the number of particles approaching the wall is

$$N = \rho \mathrm{d}V = L^2 v\mathrm{d}t\cos\theta nf(v)\mathrm{d}v\frac{1}{2}\sin\theta \mathrm{d}\theta$$

Suppose the velocity of particles after hitting the insulated wall is v_b. Then, by momentum conservation in the parallel direction of the wall, impulse on the box per collision is:

$$m\left(v\cos\theta + v_b\sqrt{1-\frac{v^2}{v_b^2}\sin^2\theta}\right) \approx m(v\cos\theta + v_b)$$

Thus, the net impulse imparted for collisions at speed v and at angle θ is:

$$\mathcal{I}^* = Nm(v\cos\theta + v_b) = \frac{1}{2}nL^2 v f(v)\mathrm{d}v(v_b\cos\theta + v)\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}t$$

We can then find the force, and hence our acceleration, imparted on the wall by integrating over impulse and dividing by unit time:

$$\begin{aligned}
\mathcal{F}^* &= \frac{1}{2}nL^2\int_0^{\frac{\pi}{2}}\int_0^{\infty} v(v\cos\theta + v_b)f(v)\mathrm{d}v\sin\theta\cos\theta\mathrm{d}\theta \\
&= \frac{1}{2}nL^2\int_0^{\frac{\pi}{2}}(\langle v^2\rangle\cos\theta + \langle vv_b\rangle)\sin\theta\cos\theta\mathrm{d}\theta \\
&= \frac{1}{6}mnL^2\langle v^2\rangle + \frac{1}{4}mnL^2\langle vv_b\rangle
\end{aligned}$$

For the insulated wall, $v_b = v$, so

$$\mathcal{F} = \frac{1}{3}mnL^2\langle v^2\rangle.$$

The net force imparted on the box is thus

$$\sum\mathcal{F} = \frac{1}{6}mnL^2\langle v^2\rangle\left[\frac{3}{2}\frac{\langle vv_b\rangle}{\langle v^2\rangle} - 1\right]$$

Since $\langle vv_b\rangle / \langle v^2\rangle \approx \frac{\sqrt{T_0T_b}}{T_0} = \sqrt{\frac{T_b}{T_0}}$, we can write our final answer as

$$a = \frac{1}{6}nL^2\frac{3k_BT_0}{m}\left(\frac{3}{2}\sqrt{\frac{T_b}{T_0}} - 1\right) = \frac{nL^2k_BT_0}{2m}\left(\frac{3}{2}\sqrt{\frac{T_b}{T_0}} - 1\right).$$

Remarks on the Net Momentum of the System

Though analyzing the impulse imparted at the conductive and insulated walls clearly indicates that the box moves from its initial position, most solvers, including the problem writer, overlooked the rather obvious fact that the net force on the system is zero. If we set the system to include the box, the gas, and the heating element, all interactions that occur in the consequent motion of the box are undoubtedly adiabatic; in other words, it doesn't make physical sense for the final momentum of the system to be nonzero – the box's COM **must** return to rest, obeying the laws of Newtonian mechanics. Let's divide the problem into two cases based on the size of L relative to the gas's mean free path λ.

$L \ll \lambda$

$L \ll \lambda$ implies that particles interact solely with the walls and not with themselves – once a particle collides with the conductive wall, gains momentum, and moves straight towards the insulated wall and collides with it. It's pretty easy to imagine that the box won't move all that much from its original position. In the first few collisions, the box will surely be accelerated towards the conductive wall, but it will come to a stop when the "hot" particles collide with the insulated wall, delivering their momentum. Once all particles collide with the conductive wall, the collision frequency and average impulse per collision at the two walls will be the same, resulting in zero net force.

$L \gg \lambda$

For this case, particles do collide with each other, and it is possible for the box to be accelerated for a sustained period of time. The temperature of the gas will gradually increase from the conductive side through gas particle collisions, eventually reaching thermal equilibrium at T_b.
The box will certainly accelerate from its initial position. There are multiple factors that causes the box to slow down, as it has been noted by several solvers. For instance, when the box's speed becomes greater than the average velocity of gas particles, there will be a rapid decline in the collision rate of gas particles at the conductive wall. This is supported by some quick calculations.

When the box is moving with velocity v_b, particles that collide with the conductive wall have a horizontal component of velocity greater than or equal to v_b. The number of particles per area that collide with the conductive wall can therefore be found as:

$$\frac{1}{2} n_b \int_0^{\frac{\pi}{2}} \left(\int_{v_b \sec\theta}^{\infty} v f(v) dv \right) \sin\theta \cos\theta d\theta$$

where f(v) is the Maxwell distributuon function of the gas in the vicinity of the conductive wall. If we evaluate the integral for different values of v_b, we find that the collision frequency per unit area decreases rapidly as v_b exceeds the average velocity of gas particles $\langle v \rangle$. For instance, collision frequency per unit area decreases from its initial value by a factor of 1000 when $v_b = 2\langle v \rangle$.

This effect is also accompanied by "weaker" collisions at the conductive wall when considering particle velocities relative to the box, ultimately causing the box to slow down. But *howmuch* does the box's speed decrease by? It would slow down until $v = v_{\text{critical}}$ and the pressure on the conductive wall overcomes than that on the insulated wall again. Then, the box will accelerate towards the conductive wall and the cycle repeats. However, this won't continue indefinitely. There are several ways to justify this as well. For instance, as the gas in the vicinity of the insulated wall heats up, the maximum pressure difference (so when the velocity of the box is at minimum) of the conductive and insulated walls $P_c - P_i$ will decrease. This means two things:

1. The maximum speed of the box between two consecutive "slow-downs" will get smaller

2. The $v_{critical}$ necessary for $P_c > P_i$ will also get smaller

Eventually, when the entire gas reaches T_b, v_{critical} will just be zero, so the box will eventually come to a full stop, as needed. From the two observations above, we can conclude that the box will eventually return to rest after its velocity-time curve undergoes something that resembles damped harmonic oscillations.

T5: Quantum Computing

In this problem, you will learn the fundamentals of quantum computers, as well as the physics on how they can be constructed! We have tried to provide as much background information as necessary, but if you believe some part is missing or unclear, please fill out the clarifications form.

Introduction

Physicists use *braket* notation to describe vectors in quantum systems. When using a vector $\vec{v}$ to describe a quantum state, the *ket*, written as $|v\rangle$ can be used. Both notations below are equivalent:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The *bra*, on the other hand, is the conjugate transpose of the ket $\langle v| = (|v\rangle)^\dagger$. Given two vectors $|v\rangle$ and $|w\rangle$, the *braket* $\langle v|w\rangle = |v\rangle \cdot |w\rangle$ is the inner product of both vectors. This notation will be used throughout this problem.

In any digital device, information is communicated via 0s and 1s, or binary code. The simplest units of this information are called bits. Similar to a bit, the *qubit* can be represented as a linear combination of two orthogonal states: quantum-0 and quantum-1, which are typically $|0\rangle$ and $|1\rangle$. Here,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Typically, we write a single qubit state as

$$|\Psi\rangle = a\,|0\rangle + b\,|1\rangle\,,$$

where $a, b \in \mathbb{C}$, and $\langle\Psi\rangle\,\Psi = 1$.

(a) **(1 pt.)** A qubit is prepared in the state $a\,|0\rangle + b\,|1\rangle$.

(i) What is the probability of measuring the qubit in the state $|0\rangle$?

(ii) What is the probability of measuring the qubit in the state $|-\rangle = |0\rangle - |1\rangle$?

Hint: If you are still confused about measurement (it's tricky!), check out this qiskit article. You can ignore all the parts with code, we'll save those for the computer science students writing OCSO.

A quantum gate performs an unitary operator on a quantum state. Applying an operator (sometimes known as a gate) to a qubit state can be represented in the diagram below.

$$|\Psi\rangle \quad \text{———}\boxed{\hat{U}}\text{———} \quad \hat{U}\,|\Psi\rangle$$

where $\hat{U}$ is a local unitary since it only acts on a single qubit. There are five important gates:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here, I, X, Y, Z form the four **Pauli matrices** and H is known as the **Hadamard** gate, which we will use later on when we talk about entanglement.

For example, if $|\Psi\rangle = 0.6\,|0\rangle + 0.8\,|1\rangle$ and apply the gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we end up with

$$X\,|\Psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix} = 0.8\,|0\rangle + 0.6\,|1\rangle\,.$$

(b) **(1 pt.)** A qubit is prepared in the state $|\Psi\rangle = a|0\rangle + b|1\rangle$. What is the probability of measuring the qubit $\hat{U}|\Psi\rangle$ in the state $|0\rangle$? Express your answer in terms of a, b, and properties of the unitary $\hat{U}$.

The heart of quantum information lies in what we can do with more than a single qubit. If one qubit has two dimensions ($|0\rangle$ and $|1\rangle$), then a two-qubit system can be represented in four dimensions. For a two qubit system, the state can be written as $a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$, where $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ can be seen as the "basis vectors." If we have two independent qubits, i.e. $|\Psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\Psi_2\rangle = c|0\rangle + d|1\rangle$, we can represent their combined state using the **tensor product**, i.e.

$$\begin{aligned}|\Psi\rangle &= |\Psi_1\rangle \otimes |\Psi_2\rangle \\ &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle.\end{aligned}$$

Here, we can see that

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Note that not every two qubit state can be written as a tensor product. When this occurs, we say that they are **entangled.** We can immediately determine if a state is entangled by calculating its **concurrence,** defined by

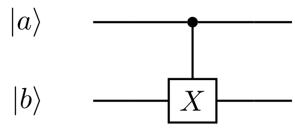
$$C = 2|a_0a_3 - a_1a_2|.$$

If $C = 0$, then the two qubits are separate and the system is **separable.** If $C = 1$, the system is maximally entangled, such as

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

Physically, this means that a measurement of one qubit directly leads to a "collapse" of the other qubit (this is the classic example shown in popular science media). Note that $0 \le C \le 1$.

We can change the concurrence using a **control operation.** For example,



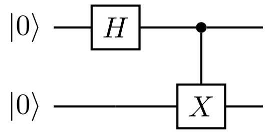
performs the **CNOT** gate. The unitary X is applied to $|b\rangle$ if $|a\rangle = 1$, otherwise nothing is done. That is, we have:

$$\begin{aligned}|00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle.\end{aligned}$$

The CNOT gate is an example of a global unitary, since it acts on more than one qubit. Global unitaries for 2 qubit systems can be written as a 4×4 matrix. For example, we can write

$$CNOT = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

where $|00\rangle, \ldots, |11\rangle$ form the 4 standard basis vectors. We can combine local and global unitaries to create entangled states. For example, consider the following circuit:



The initial state is $|\Psi_{\text{in}}\rangle = |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$. After applying the Hadamard gate H, the state becomes

$$|\Psi_{\text{middle}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}.$$

After applying the CNOT gate, the state becomes:

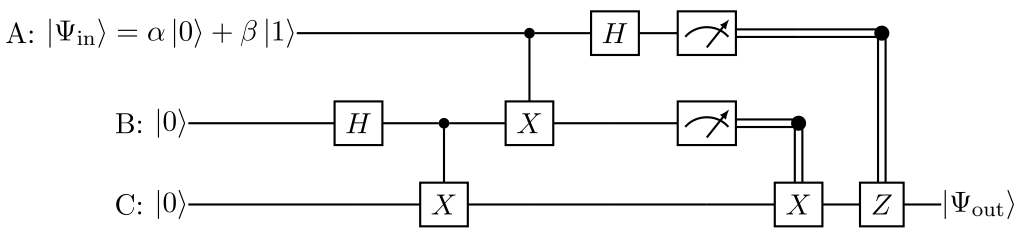
$$|\Psi_{\text{out}}\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

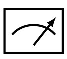
Note that we can avoid matrix multiplication in this last step by seeing what CNOT does on each term of $|\Psi_{\text{middle}}\rangle$. CNOT will not have an effect on $\frac{1}{2}|00\rangle$ since the first qubit is $|0\rangle$. CNOT will have an effect on $\frac{1}{2}|10\rangle$ since the first qubit is a $|1\rangle$, so it'll flip the second qubit to a $|1\rangle$, giving us the map $\frac{1}{2}|10\rangle \mapsto \frac{1}{2}|11\rangle$.

(c) **(1 pt.)** Construct a quantum circuit where the input state is $|00\rangle$ and the output state is $\frac{i}{\sqrt{2}}(|0\rangle - |1\rangle)$ using only $X, Y, Z, H, CNOT$ gates.

Quantum Teleportation

Quantum teleportation is the transfer of the quantum state of one qubit to another (not the actual physical qubit) using a shared entangled resource and two classical bits of information. It is performed using the following circuit.



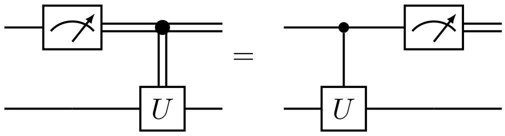
The  gate measures the qubit (returns either a 0 or a 1) and the wider wire represents that information that flows through this wire is a classical bit.

(d) **(1 pt.)** Verify that the above circuit does teleport the qubit from the top branch to the bottom branch by looking at the specific case of $\alpha = \beta = \frac{1}{\sqrt{2}}$

(e) **(3 pts.)** After the first operation is performed on branch C, the branch is brought a very far distance from the other two branches. By doing so, it appears we can create faster-than-light communication during the teleportation process, which is impossible! Explain why there is no contradiction. Justify rigorously.

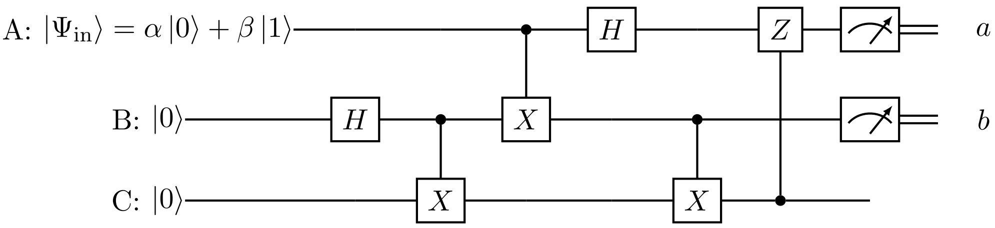
We can analyze this by performing matrix multiplication, but using a circuit-based approach is much cleaner. To do so, we need to use the **Griffiths-Niu Theorem.**

(f) **(2 pts.)** The following circuits, according to the Griffiths-Niu Theorem, are equivalent:

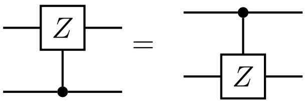


Prove the Griffiths-Niu Theorem.

Using this theorem, we can redraw our circuit as:

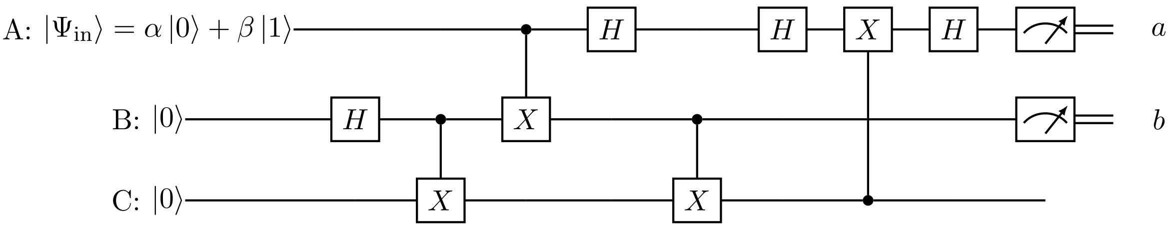


(g) **(1 pts.)** For a control-Z gate, it doesn't matter which branch is the control. In other words,

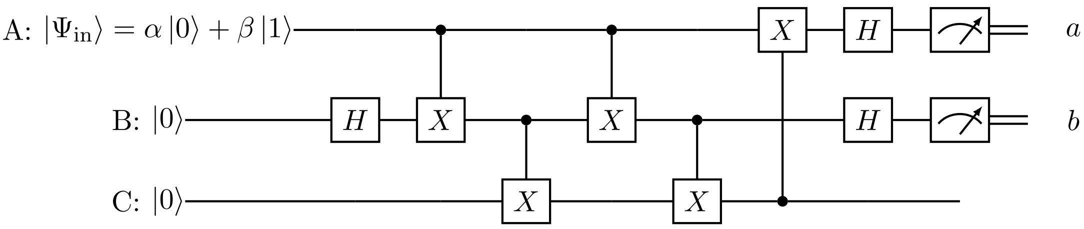


Prove this relationship.

Using the above problem, we can flip the control-Z gate. Then using the identity $Z = HXH$, we can reduce it further:

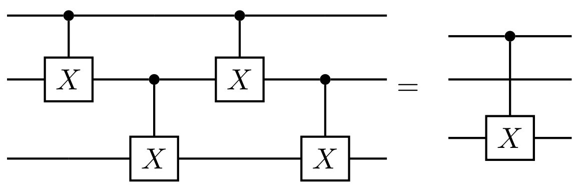


Since $H^2 = I$, we can simplify the top part. Furthermore, we can introduce another CNOT between the first and the second branch.

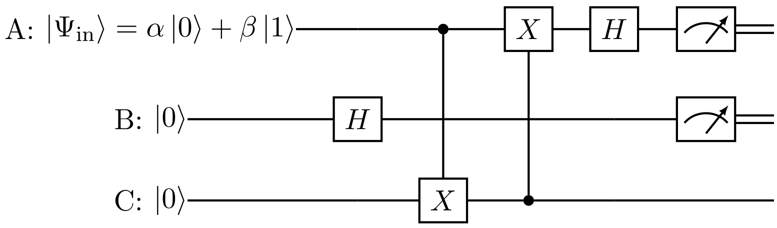


We were allowed to introduce this CNOT gate since $XH|0\rangle = H|0\rangle$. This actually makes it easier using the following problem:

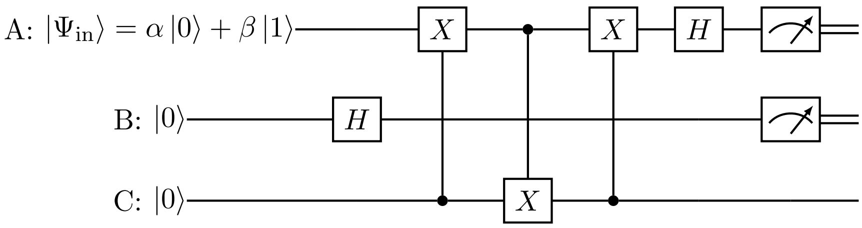
(h) **(2 pts.)** Prove that the below two circuits are equivalent.



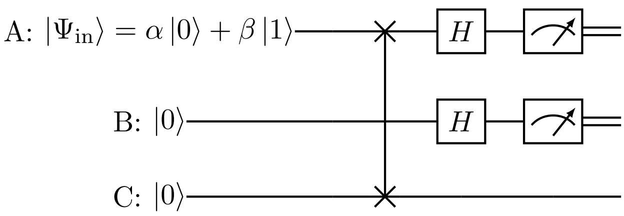
Using this substitution, we end up with:



We can now introduce another CNOT gate, which doesn't do anything since C will always be $|0\rangle$.



Three alternating CNOT gates is equivalent to the SWAP gate, so we can write:



where we clearly see a swapping that occurs between the top and bottom branch!

Building Quantum Computers

According to theoretical physcist David P. Divencenzo, there are five necessary (but not necessarily sufficient) criteria to build a quantum computer:

- A well-characterized qubit.
- The ability to initialize qubits.
- Long and relevant decoherence times.
- A "universal set" of quantum gates.
- The ability to measure qubits.

In this section, we will focus on how we can create qubits and how we can create a universal set of quantum gates. Consider two energy levels E_1, E_0 as the qubit states $|1\rangle, |0\rangle$ respectively. Assume that

$$E_1 = \frac{1}{2}\hbar\omega, \qquad E_0 = -\frac{1}{2}\hbar\omega.$$

Also assume that the qubit state is time varying, in the form of:

$$|\Psi(t)\rangle = A(t)|0\rangle + B(t)|1\rangle .$$

(i) **(14 pts.)** Using the above setup, show how we can implement the quantum gates X, Y, Z. *Hint:* The Schrodinger Equation tells us

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{\mathcal{H}}|\Psi(t)\rangle ,$$

where $\hat{\mathcal{H}} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$.

Grading Scheme

(a) (0.5 pts) $|a|^2$

(b) Three answers were acceptable due to question ambiguity:

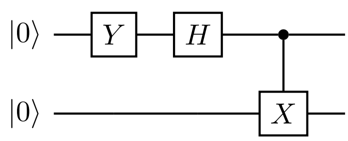
  (i) (0.5 pts) 0

  (ii) (0.5 pts) $\frac{1}{2}|a-b|^2$

  (iii) (0.5 pts) $|a-b|^2$

(c) (1 pts) $|aU_{11} + bU_{12}|^2$

(d) (1 pts) Drawing/describing the below or equivalent:



To check equivalent, use website.

(e) (1 pts) Via either matrix multiplication or circuit analysis (or equivalent)

Note: The last step is the trickiest (where you have to deal with measurement, resulting in 4 cases). Make sure the participant checks all 4 cases or uses a clever method to cirumvent checking the cases. If they skip over the measurement step (or it's not clearly written), deduct 0.5 pts.

(f) (1 pts) Recognizing that classical bits need to be communicated via control wires so the qubit isn't teleported faster than light.

(1 pts) Stating that *nothing* can be recovered from the state before classical bits are communicated.

(1 pts) Proving the above statement. (This actually turns out to be much harder, so if the team attempts to prove the above, then give the point. The main idea is to recognize that simply stating you need control wires to prevent FTL communication is not sufficient)

(g) (2 pts) Complete proof. Partial proofs will receive 1 point.

(h) (1 pts) Complete proof. Partial proofs will receive 0.5 points.

(i) (1 pts) Complete proof. Partial proofs will receive 0.5 points.

(j)
- (1 pts) Sets up the system of ODE
- (2 pts) Solves the ODE
- (6 pts) Recognizes that the solution can be written in terms of Pauli matrices,

$$|\Psi(t)\rangle = \left(\cos(\omega t/2)\hat{I} + i\sin(\omega t/2)\hat{Z}\right)|\Psi(0)\rangle$$

- (5 pts) Shows how using the above form, we can create the X, Y, Z gates.